INFLUENCE OF THERMAL RADIATION ON THE LAMINAR BOUNDARY LAYER OF A NONABSORBING FLUID

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The influence of thermal radiation on the laminar boundary layer of a nonabsorbing fluid with variable thermophysical properties flowing around a heat emitting surface is investigated under natural and forced convection conditions.

The interaction between thermal radiation and convection originates upon assigning a boundary condition of the second kind on a heat radiating surface over which a fluid flows. It causes a transition from heat exchange under a boundary condition of the second kind to heat exchange with a boundary condition of the first kind along the length of the surface [1].

Of the investigations touching upon this problem, the papers [1, 2] should be noted. A solution of this problem has been obtained in [2] for a heat insulated surface (an absolutely non-heat-conducting plate) in an air flow with compressibility and dissipation taken into account. An analysis is carried out in [1] and solutions are obtained for the interaction between thermal radiation and natural and forced convection. The solutions obtained are valid only for limit cases: small and large values of the radiation parameter ξ. Solutions for a flat surface are obtained below in a first approximation by the method of averaging functional corrections [3], which permit determination of the temperature and convective heat exchange of a heat emitting surface under forced and natural convection conditions for a boundary condition of the second kind.

Natural Convection. The stationary laminar, natural convection of a nonabsorbing fluid around a vertical flat plate heated by a constant heat flux q_w is considered. Heat transmission is accomplished during the interaction of radiant heat exchange (to the surrounding medium of temperature T_e) with the convection (to the wetting fluid with temperature T_0). Let us consider the fluid to be a perfect gas ($\rho T = \rho_0 T_0$), the Prandtl number and specific heat to be constants, and the coefficients of viscosity and heat conductivity to depend linearly on the temperature ($\mu/\mu_0 = \lambda/\lambda_0 = bT/T_0$). Let us write the mass, momentum, and energy conservation equations in Dorodnitsyn variables $\xi(x)$, $\eta(x, y)$ [4] by neglecting dissipation:

$$\frac{\partial u}{\partial \xi} + \frac{\partial V}{\partial \eta} = 0 \quad \text{or} \quad V = -\int_0^\eta \frac{\partial u}{\partial \xi}\, d\eta, \tag{1}$$

$$u\frac{\partial u}{\partial \xi} + V\frac{\partial u}{\partial \eta} = b\nu_0 \frac{\partial^2 u}{\partial \eta^2} + g\beta(T - T_0), \tag{2}$$

$$u\frac{\partial T}{\partial \xi} + V\frac{\partial T}{\partial \eta} = ba_0 \frac{\partial^2 T}{\partial \eta^2} \tag{3}$$

and the boundary conditions under the assumption that the thicknesses of the thermal and hydrodynamic boundary layers are equal:

$$\eta = 0 \quad \frac{\partial T}{\partial \eta} = -\frac{q_k}{\lambda_0 b} = -\frac{q_w - \sigma\varepsilon(T_w^4 - T_e^4)}{\lambda_0 b}, \tag{4}$$

$$u = V = 0, \tag{5}$$

$$\eta \geqslant \delta \quad \frac{\partial T}{\partial \eta} = 0, \quad T = T_0, \tag{6}$$

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$$\frac{\partial u}{\partial \eta} = 0, \quad u = 0. \tag{7}$$

In order to find the solutions of the system (1)-(7) in a first approximation by the Yu. D. Sokolov method of averaging functional corrections, let us write (3) and (2) as [3]

$$f_1(\xi) = ba_0 \frac{\partial^2 T}{\partial \eta^2} \tag{8}$$

and

$$f_2(\xi) = b\nu_0 \frac{\partial^2 u}{\partial \eta^2} + g\beta (T - T_0), \tag{9}$$

where

$$f_1(\xi) = \frac{1}{\delta} \int_0^\delta \left\{ u \frac{\partial T}{\partial \xi} - \frac{\partial T}{\partial \eta} \int_0^\eta \frac{\partial u}{\partial \xi} d\eta \right\} d\eta \tag{10}$$

and

$$f_2(\xi) = \frac{1}{\delta} \int_0^\delta \left\{ u \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \int_0^\eta \frac{\partial u}{\partial \xi} d\eta \right\} d\eta. \tag{11}$$

After having determined the constant of integration and $f_1(\xi) = a_0 q_k/\lambda_0 \delta$ from the boundary conditions (4), (6), the integral of (8) yields the temperature profile

$$T - T_0 = \frac{q_k \delta}{2\lambda_0 b} \left(1 - \frac{\eta}{\delta}\right)^2. \tag{12}$$

For $\eta = 0$ it follows from (12) that

$$\delta = \frac{2\lambda_0 b (T_w - T_0)}{q_k} = \frac{2\lambda_0 b}{\sigma\varepsilon T_0^3} \cdot \frac{(\theta_w - 1)}{(\theta_l^4 - \theta_w^4)}. \tag{13}$$

Analogously taking account of (12), we find the velocity profile from (9):

$$u = \frac{g\beta q_k \delta^3}{6\lambda_0 \nu_0 b^2} \left(- \frac{\eta^4}{4\delta^4} + \frac{\eta^3}{\delta^3} - \frac{5\eta^2}{4\delta^2} + \frac{\eta}{2\delta} \right). \tag{14}$$

Substituting the values u, $\partial T/\partial\xi$, $\partial T/\partial\eta$, $\partial u/\partial\xi$ and $f_1(\xi)$ into (10), we obtain after integrating

$$\frac{g\beta}{840\lambda_0^2 \nu_0 b^3} \cdot \frac{d}{d\xi} q_k^2 \delta^5 = \frac{a_0 q_k}{\lambda_0}, \tag{15}$$

which we write taking account of (13) as

$$\frac{5(\theta_w - 1)^4 d\theta_w}{(\theta_l^4 - \theta_w^4)^4} + \frac{3(\theta_w - 1)^5 d\theta_w^4}{(\theta_l^4 - \theta_w^4)^5} = \frac{105(\sigma\varepsilon T_0^3)^4 a_0 \nu_0 d\xi}{4g\beta\lambda_0^4 b^2 T_0}. \tag{16}$$

The integral of the differential equation (16) for a boundary condition on the leading edge of the plate $\theta_w(\xi = 0) = 1$ yields a transcendental equation to compute the temperature change along the plate length

$$\frac{3}{5} \frac{(\theta_w - 1)^5}{(\theta_l^4 - \theta_w^4)^4} + G(\theta_l;\ \theta_w) - G(\theta_l;\ 1) = \frac{21\nu_0 a_0 \xi (\sigma\varepsilon T_0^3)^4}{g\beta\lambda_0^4 b^2 T_0}, \tag{17}$$

and therefore the convective heat exchange also

$$\mathrm{Nu}_\xi = \frac{q_k \xi}{\lambda_0 (T_w - T_0)} = \frac{1}{\sqrt[4]{\frac{35b^2}{\mathrm{Ra}_\xi} - \frac{5}{3(\theta_w - 1)} \left(\frac{\lambda_0}{\xi\sigma\varepsilon T_0^3} \right)^4 [G(\theta_l;\ \theta_w) - G(\theta_l;\ 1)]}}, \tag{18}$$

where

$$G(\theta_l;\ \theta_w) = \int \frac{(\theta_w - 1)^4 d\theta_w}{(\theta_l^4 - \theta_w^4)^4} = \frac{\theta_w (\theta_w - 1)^4}{12\theta_l^4 (\theta_l^4 - \theta_w^4)^3} + (21\theta_l^8\theta_w + 7\theta_l^4\theta_w^5$$

$$- 128\theta_l^8 + 486\theta_l^4\theta_w^3 - 270\theta_w^7 - 400\theta_l^4\theta_w^2 + 240\theta_w^6 + 121\theta_l^4\theta_w - 77\theta_w^5)/384\theta_l^{12}(\theta_l^4 - \theta_w^4)^2$$

$$+ \frac{(77 + 270\theta_l^2 - 7\theta_l^4)\ln\left|\frac{\theta_l + \theta_w}{\theta_l - \theta_w}\right| + (154 - 540\theta_l^2 - 14\theta_l^4)\operatorname{arctg}\frac{\theta_w}{\theta_l}}{512\theta_l^{15}} - \frac{5}{16\theta_l^{14}}\ln\left|\frac{\theta_l^2 + \theta_w^2}{\theta_l^2 - \theta_w^2}\right|. \tag{19}$$

To analyze the equation obtained, let us determine the relative contribution of the term $G(\theta_l;\ \theta_w) - G(\theta_l;\ 1)$ in (17) for the limit cases: small ($x \to 0$; $\theta \to 1$) and large (x large; $\theta_w \to \theta_l$ since $q_k \to 0$) boundary layer thicknesses:

$$\lim \frac{\int_1^{\theta_w} \frac{(\theta_w - 1)^4 d\theta_w}{(\theta_l^4 - \theta_w^4)^4}}{\frac{3}{5}\frac{(\theta_w - 1)^5}{(\theta_l^4 - \theta_w^4)^4}} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases} = \begin{cases} \frac{1}{3} & (\theta_w \to 1) \\ 0 & (\theta_w \to \theta_l). \end{cases} \tag{20}$$

The relationship (20) permits representation of (18) as

$$\mathrm{Nu}_\xi = 0.442\sqrt{b}\,\mathrm{Ra}^{0.25}, \quad \theta_w \to 1 \quad (x \to 0) \tag{18a}$$

and

$$\mathrm{Nu}_\xi = 0.409\sqrt{b}\,\mathrm{Ra}^{0.25}, \quad \theta_w \to \theta_l \quad (x \text{ large}). \tag{18b}$$

Therefore, (18) describes the transition from heat exchange under boundary conditions of the second kind (18a) to heat exchange with a boundary condition of the first kind (18b) expressed by the presence of thermal radiation. The differences between the limit solutions and the exact values are not more than 4% [1].

Forced Convection. The stationary flow of a nonabsorbing fluid around a flat surface heated by a constant heat flux q_w is considered. Heat transmission is accomplished during interaction between the radiant heat exchange (to the surrounding medium of temperature T_e) and the convection (to the wetting fluid with temperature T_0).

The problem in Dorodnitsyn variables is described mathematically by the mass, momentum, and energy conservation equations:

$$\frac{\partial u}{\partial \xi} + \frac{\partial V}{\partial \eta} = 0, \tag{21}$$

$$u\frac{\partial u}{\partial \xi} + V\frac{\partial u}{\partial \eta} = b\nu_0\frac{\partial^2 u}{\partial \eta^2}, \tag{22}$$

$$u\frac{\partial T}{\partial \xi} + V\frac{\partial T}{\partial \eta} = ba_0\frac{\partial^2 T}{\partial \eta^2} + \frac{\nu_0 b}{c_p}\left(\frac{\partial u}{\partial \eta}\right)^2 \tag{23}$$

under the boundary conditions

$$\eta = 0 \quad \frac{\partial T}{\partial \eta} = -\frac{q_w - \sigma\varepsilon(T_w^4 - T_e^4)}{\lambda_0 b} = -\frac{q_k}{\lambda_0 b}, \tag{24}$$

$$u = V = 0, \tag{25}$$

$$\eta \gg \delta_m \quad \frac{\partial T}{\partial \eta} = 0; \quad T = T_0, \tag{26}$$

$$\eta \gg \delta_g \quad \frac{\partial u}{\partial \eta} = 0; \quad u = U_0. \tag{27}$$

The solution of the hydrodynamic problem (21), (22), (25), (27) in a first approximation by the Yu. D. Sokolov method yields the following expressions for the velocity profile and the boundary layer thickness:

$$u = U_0 \frac{\eta}{\delta_g}\left(2 - \frac{\eta}{\delta_g}\right) \text{ and } \delta_g = \sqrt{\frac{30\nu_0 b\xi}{U_0}}. \tag{28}$$

The solution of (23) by the Yu. D. Sokolov method can be obtained only in the following cases: taking account of dissipation for Pr = 1 or for a heat insulated surface without radiation without taking account of dissipation in the presence of radiation for any Pr.

Solving (21), (23) without taking account of dissipation under the boundary conditions (24), (26), we obtain the temperature profile

$$T - T_0 = \frac{q_k \delta_m}{2\lambda_0 b}\left(1 - \frac{\eta}{\delta_m}\right)^2, \tag{29}$$

which for $\eta = 0$ goes over into the relationship

$$T_w - T_0 = \frac{q_k \delta_m}{2\lambda_0 b} \quad \text{or} \quad \delta_m = \frac{2\lambda_0 b}{\sigma\varepsilon T_0^3} \cdot \frac{(\theta_w - 1)}{(\theta_l^4 - \theta_w^4)} \tag{30}$$

and the differential equation

$$\frac{d}{d\xi} q_k \delta_m^2 H(\Delta) = \frac{6ba_0 q_k}{U_0}, \tag{31}$$

where

$$H(\Delta) = \frac{\Delta}{2} - \frac{\Delta^2}{10} \quad \text{for} \quad \Delta = \frac{\delta_m}{\delta_g} < 1$$

and

$$H(\Delta) = 1 - \frac{1}{\Delta} + \frac{1}{2\Delta^2} - \frac{1}{10\Delta^3} \quad \text{for} \quad \Delta > 1.$$

Equation (31), as (15), describes the transition from heat exchange under a boundary condition of the second kind to heat exchange with a boundary condition of the first kind along the plate length, and can be solved only numerically in the general case. To construct its approximate solution, let us replace the ratio $\Delta = \delta_m/\delta_g$, equal to $Pr^{-1/3}$ for T_w = const (the solution (29) with q_k replaced by $T_w - T_0$ according to (30)) and 0.78 $Pr^{-1/3}$ for q_k = const (the solution (29) for q_k = const) by the mean value Δ = 0.89 $Pr^{-1/3}$ and by taking account of (30) let us reduce (31) to a form analogous to (16):

$$\frac{2(\theta_w - 1)\,d\theta_w}{(\theta_l^4 - \theta_w^4)^2} + \frac{(\theta_w - 1)^2 d\theta_w^4}{(\theta_l^4 - \theta_w^4)^3} = \left(\frac{\sigma\varepsilon T_0^3}{2\lambda_0}\right)^2 \frac{6a_0 d\xi}{U_0 b H\left(0.89\,\mathrm{Pr}^{-\frac{1}{3}}\right)}. \tag{32}$$

Integrating the equation obtained under a boundary condition on the leading edge of the plate $\theta_{w(\xi = 0)} = 1$, we obtain relationships to compute the change in temperature and convective heat exchange along the length of the surface:

$$\frac{1}{2} \frac{(\theta_w - 1)^2}{(\theta_l^4 - \theta_w^4)^2} + F(\theta_l;\ \theta_w) - F(\theta_l;\ 1) = \left(\frac{\sigma\varepsilon T_0^3}{2\lambda_0}\right)^2 \frac{6a_0\xi}{U_0 b H\left(0.89\,\mathrm{Pr}^{-\frac{1}{3}}\right)} \tag{33}$$

and

$$\mathrm{Nu}_\xi = \frac{q_k \xi}{\lambda_0 (T_w - T_0)} = \left[\frac{3b}{H\left(0.89\,\mathrm{Pr}^{-\frac{1}{3}}\right)\mathrm{Pe}_\xi} - \frac{1}{2}\left(\frac{2\lambda_0}{\sigma\varepsilon T_0^3 \xi}\right)^2 \left[F(\theta_l;\theta_w) - F(\theta_l;1)\right]\right]^{-1/2}, \tag{34}$$

where

$$F(\theta_l;\ \theta_w) = \frac{\theta_w(\theta_w - 1)}{4\theta_l^4(\theta_l^4 - \theta_w^4)} + \frac{1}{8\theta_l^6}\ln\left|\frac{\theta_l^2 + \theta_w^2}{\theta_l^2 - \theta_w^2}\right| - \frac{3}{16\theta_л^7}\left[\ln\left|\frac{\theta_l + \theta_w}{\theta_l - \theta_w}\right| + 2\,\mathrm{arctg}\,\frac{\theta_w}{\theta_l}\right]. \tag{35}$$

Using a relation analogous to (20)

$$\lim \frac{\int\limits_{1}^{\theta_w} \frac{(\theta_w - 1)\, d\theta_w}{(\theta_l^4 - \theta_w^4)^2}}{\frac{1}{2}\,\frac{(\theta_w - 1)^2}{(\theta_l^4 - \theta_w^4)^2}} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \end{cases} = \begin{cases} 1 & \theta_w \to 1 \\ 0 & \theta_w \to \theta_l, \end{cases} \tag{36}$$

we obtain the limit solutions

$$\mathrm{Nu}_\xi = \sqrt{H\left(0.89\,\mathrm{Pr}^{-\frac{1}{3}}\right)\frac{2}{3}\, b\, \mathrm{Pe}_\xi^{0.5}} \quad (\theta_w \to 1) \tag{34a}$$

and

$$\mathrm{Nu}_\xi = \sqrt{H\left(0.89\,\mathrm{Pr}^{-\frac{1}{3}}\right)\frac{1}{3}\, b\, \mathrm{Pe}_\xi^{0.5}} \quad (\theta_w \to \theta_l), \tag{34b}$$

which differ by 5% from the exact values [1].

In conclusion, let us note that the solution of (23) in the presence of radiation and taking account of dissipation agrees for $\mathrm{Pr} = 1$ with (33) and (34), and the absolute temperature T in the expression (29) for the temperature profile is replaced by $T_* = T + u^2/2c_p$.

NOTATION

T_w, T_e, T_0	are the absolute temperatures of the wall, surrounding medium, and wetting fluid, respectively;
$\theta_w = T_w/T_0$	is the dimensionless wall temperature;
$\theta_l = \sqrt[4]{(q_w + \sigma\varepsilon T_e^4)/\sigma\varepsilon T_0^4}$	is the dimensionless wall temperature in the case of only radiation heat exchange;
δ, δ_g, δ_m	are the boundary layer thickness, and hydrodynamic and thermal boundary layer thickness;
q_w, q_k	are the wall and convective heat flux, respectively;
$\beta = 1/T_0$	is the coefficient of thermal expansion;
σ	is the Stefan–Boltzmann constant;
$\mathrm{Ra}_\xi = g\beta_\xi^0(T_w - T_0)/\nu_0 a_0$	is the Rayleigh criterion;
$\mathrm{Pe}_\xi = U_0\xi/a_0$	is the Peclet criterion;
$\xi = x$, $\eta = 1/\rho_0 \int_0^y \rho dy$	are the Dorodnitsyn variables.

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